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LETTER TO THE EDITOR

Determinantal solution of the logistic map

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Abstract. The solution of the logistic map at the generic nth iteration is given in terms of the initial datum and the parameter as the characteristic polynomial of a simple matrix.

The relevance of the logistic (quadratic) map (LM) is well known: it has a very large spectrum of applications in various fields (physics, biology, economy, fractal geometry); moreover, as the simplest nonlinear discrete-time dynamical system, it is also very important from a theoretical point of view for providing the simplest route to chaos [1]. In this paper we will show that the solution of the LM can be written in a closed form, indeed it can be given as the characteristic polynomial of a simple matrix that is explicitly written for any iterate in terms of the parameter and the initial datum of the LM itself.

We choose the following standard form for the LM:

$$z_{n+1} = z_n^2 + c \tag{1}$$

where the 'dynamical' variable z_n as well the parameter c can be complex; we will denote by λ the initial datum:

$$z_0 = \lambda.$$

The other standard form for the LM, namely

$$x_{n+1} = \mu\ x_n(1 - x_n) \tag{2}$$

can be easily obtained from (1) via a simple linear transformation.

To complete the notational setting, in the following an indexed capital letter, say $\mathbf{A}^{(m)}$, will denote the $2^m \times 2^m$ square matrix $\mathbf{A}$, $\mathbf{I}^{(m)}$ being the $2^m \times 2^m$ identity matrix ($\mathbf{I}^{(0)} = 1$).

It is clear that the nth iteration of LM yields z_n as an even monic polynomial of the 2^n degree in λ with polynomial coefficients in the parameter c. The main aim of this letter is to prove that z_n is the characteristic polynomial of the following matrix $\mathbf{M}^{(n)}$:

$$z_n = \det(\mathbf{M}^{(n)} - \lambda \mathbf{I}^{(n)}) \qquad n = 1, 2, \ldots \tag{3}$$

$$\mathbf{M}^{(n)} = \begin{pmatrix} 0 & -\mathbf{I}^{(n-1)} \\ \mathbf{Y}^{(n-1)} & 0 \end{pmatrix} \qquad n = 1, 2, \ldots \tag{4}$$

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where the matrix $\mathbf{Y}$ is explicitly given (for any n) in terms of the parameter c by

$$Y^{(n)}_{2^k\cdot s,1+2^k\cdot(s-1)} = c \qquad n = 0, 1, 2, \ldots;\ k = 0, 1, 2, \ldots, n;\ s = 1, 2, \ldots, 2^{n-k} \tag{5}$$

$$Y^{(n)}_{j,j+1} = -1 \qquad n = 1, 2, \ldots;\ j = 1, 2, \ldots, 2^n - 1$$

all the other entries being equal to zero.

It is important to notice that the above explicitly defined matrix $\mathbf{Y}$ can also be constructed recursively through:

$$\mathbf{Y}^{(n+1)} = \begin{pmatrix} \mathbf{Y}^{(n)} & -\mathbf{E} \\ c\mathbf{E} & \mathbf{Y}^{(n)} \end{pmatrix} \qquad n = 0, 1, 2, \ldots \tag{6}$$

where the elementary matrix $\mathbf{E}$ has just one entry equal to one in the last row, first column, all the other entries being equal to zero. Due to this recursive construction, the (quasi-triangular) matrix $\mathbf{Y}$ has a sparse 'fractalic' structure.

Using the formula for the partitioning of the determinants [2, p 126], namely

$$\det\begin{pmatrix} A & F \\ G & B \end{pmatrix} = \det(B)\det(A - F\cdot B^{-1}\cdot G) \tag{7}$$

it is possible and useful to cast equation (3) in the alternative form

$$z_n = \det(\lambda^2\mathbf{I}^{(n-1)} + \mathbf{Y}^{(n-1)}) \qquad n = 1, 2, \ldots. \tag{8}$$

Note that the above formula makes it immediately clear that in the polynomial z_n there are only even powers of the initial datum λ.

Now we are ready to prove formula (8) by induction, namely we will prove that

$$\det(\hat{\mathbf{Y}}^{(n+1)}) = (\det(\hat{\mathbf{Y}}^{(n)}))^2 + c \tag{9}$$

where, for notational convenience, we have introduced the matrix

$$\hat{\mathbf{Y}}^{(n)} = \lambda^2\mathbf{I}^{(n)} + \mathbf{Y}^{(n)}$$

Using equation (6) and formula (7), from the left-hand side of (9) one gets

$$\det\begin{pmatrix} \hat{\mathbf{Y}}^{(n)} & -\mathbf{E} \\ c\mathbf{E} & \hat{\mathbf{Y}}^{(n)} \end{pmatrix} = \det(\hat{\mathbf{Y}}^{(n)})\cdot\det(\hat{\mathbf{Y}}^{(n)} + c\mathbf{E}\cdot(\hat{\mathbf{Y}}^{(n)})^{-1}\cdot\mathbf{E}). \tag{10}$$

Due to the quasi-triangular structure of $\mathbf{Y}^{(n)}$ (and $\hat{\mathbf{Y}}^{(n)}$), it is straightforward to see that

$$\mathbf{E}\cdot(\hat{\mathbf{Y}}^{(n)})^{-1}\cdot\mathbf{E} = (\det(\hat{\mathbf{Y}}^{(n)}))^{-1}\mathbf{E}$$

$$\det((\hat{\mathbf{Y}}^{(n)}) + c(\det(\hat{\mathbf{Y}}^{(n)}))^{-1}\mathbf{E}) = \det(\hat{\mathbf{Y}}^{(n)}) + c(\det(\hat{\mathbf{Y}}^{(n)}))^{-1}.$$

Thus the right-hand side of (10) is equal to the right-hand side of (9): the recursive formula (9) is proved. Since clearly formula (8) (or equivalently (3)) holds true for $n = 1$, by induction it holds true for any n.

The knowledge of this determinantal solution of the LM could help to understand the peculiar behaviour of the LM itself: investigation in this direction should be fruitful.

We want to stress that such a solution is hardly useful for computational purposes; however, the sparse and self-similar structure of the matrix $\mathbf{Y}^{(n)}$ involved could inspire the derivation of new properties of periodic solutions of LM and, moreover, the explicit nature of this solution could be a good starting point for a perturbative analysis.

Analogies between the structure of $\mathbf{Y}^{(n)}$ and similar matrices that are important in other fields [3] are also suggestive.

However, in our opinion, the mere fact that the solution of a famous chaotic system can be written in a closed form is an important and nice result.

Note added in proof. When the first version of this paper was written, the author was unaware of the existence of a different analytical solution of the LM [4] (the author would like to thank A Vulpiani for providing this information). This analytical solution looks quite different from the one proposed in this paper: namely, the solution x_n of LM in the standard form (2) is given by

$$x_n = \langle f_0 | \mathbf{T}^n | e_1 \rangle$$

where the (infinite) bra $\langle f_0|$ is constructed in terms of the initial datum x_0, while $\langle e_1| = (1, 0, 0, \ldots)$; the elements of the matrix $\mathbf{T}$ are given in terms of the parameter μ:

$$T_{j,k} = (-1)^{j-k} \binom{k}{j-k} \mu^k \qquad j = 1, 2, 3, \ldots; k = 1, 2, 3, \ldots.$$

Remarks.

- This matrix $\mathbf{T}$ is triangular but dense, the above matrix $\mathbf{Y}$ is quasi-triangular but sparse and self-similar; moreover, here the nth power of $\mathbf{T}$ is needed to project it on the initial datum while in our approach only the characteristic polynomial $\mathbf{Y}$ is needed in the initial datum (this could be important for computational purposes).
- However, although the matrix $\mathbf{T}$ is infinite, only the finite $2^n \times 2^n$ upper-left corner of it is needed to compute the nth power.

Of course these two different analytical solutions must be connected, even if apparently in no trivial way: the investigation of this connection should also be very interesting.

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